Unit1:

Random experiment, sample space , events, classical definition of probability and the theorems of total and compound probability based on this definition, axiomatic approach to the notion of probability, important theorems based on this approach, conditional probability and independent events, Bay's theorem. 15 marks

DEFINITION OF VARIOUS TERMS:

1. Random Experiment: An experiment which can be repeated under essentially identical conditions and whose all possible results are known in advance but whose result in any particular instance cannot be definitely predicted is called a random experiment.

Eg. If we toss a coin we may either get a head or a tail, but we cannot predict our result in a particular instance in advance. Thus tossing a coin is a random experiment.

2. Sample Space: The set of all possible outcomes of a random experiment is called the sample space. It is usually denoted by **S** or **Ω .**

Eg. The sample space for the experiment 'tossing two coins simultaneously' is

S = { HH, HT, TH, TT }

3. Event: An event is the collection of all possible outcomes of an random experiment which are favorable to a phenomenon or happening related to the experiment. Every subset of the sample space defines an event.

Eg. In the ' tossing two coins simultaneously' experiment, let A be the event of getting at least one head, then A = { HH, HT, TH} $\subseteq S$.

The event A happens if the outcomes of the underlying random experiment belongs to the subset A.

4. Exhaustive Events: A set of events A_1, A_2, ---, A_n is said to be exhaustive if every possible outcome of the underlying experiment belongs to one of the A_i's i.e. $S=\bigcup_{i=1}^{n} A_i$.

Eg. In the ' tossing two coins simultaneously' experiment the set of events { A, B } where, A is the event ' getting at least one head' and B is the event ' getting at least one tail' is exhaustive.

5.Mutually Exclusive Events: Two events are said to be mutually exclusive if the happening of any one of them prevents the happening of the other. If A and B are two mutually exclusive events then $A \cap B = \phi$.

Eg. In the above mutually exclusive events A and B, we have

A = { HH, HT, TH } and B = {HT, TH, TT }. As $A \cap B \neq \phi$, A and B are not mutually exclusive. Now if C is the event 'getting exactly two tails' then C = { TT }. Now as $A \cap C = \phi$, A and C are mutually exclusive events.

6. Independent Events: Two or more than two events are said to be independent if the happening (or non happening) of an event is not affected by the occurrence of any one of the remaining events .

Eg. If we draw a card from a pack of well-shuffled cards and replace it before drawing the 2nd card, the result of 2nd draw is independent of the first draw. But , however, if the 1st card drawn is not replaced then the 2nd draw is not independent of the 1st draw.

7.Trials: A trails is a single perfomance of a random experiment. If we toss a coin once , we get one trial of the coin tossing experiment , if we toss twice , we get two trials and so on.

CLASSICAL OR PRIORI DEFINITION OF PROBABILITY:

If a trial result in n exhaustive, mutually exclusive and equally likely cases and if m of them are favorable to the happening of an event E, then the probability p of the happening of the event E is given by p = P(E) = $\frac{m}{n}$

Clearly number of cases favorable to $\overline{E}$ is n-m, so P($\overline{E}$) = $\frac{n-m}{n}$ = 1 - P(E)

$\Rightarrow P(E) + P(\overline{E})$ = 1

Also, since

$$0 \leq m \leq n \, and \, 0 \leq n-m \leq n$$
$$\Rightarrow \frac{0}{n} \leq \frac{m}{n} \leq \frac{n}{n} \, and \, \frac{0}{n} \leq \frac{n-m}{n} \leq \frac{n}{n}$$
$$\Rightarrow 0 \leq P(E) \leq 1 \, and \, 0 \leq P(\overline{E}) \leq 1$$

Demerits: Classical definition fails if

1. Exhaustive number of cases is infinite countable or uncountable.
2. The outcomes are not equally likely.

VON MISIS'S STATISTICAL (EMPERICAL) DEFINITION OF PROBABILITY:

If a trial is repeated a number of times under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the events happens to the number of trial as the number of trials becomes indefinitely large is called the probability of happening of this event ie. if in n trials an event E occurs m times, then

$$p = P(E) = \lim_{n\to\infty} \frac{m}{n}$$

Here it is assume that the limit exists and unique. Clearly, $0 \leq p \leq 1$

Demerits: Statistical definition remove most of the deficiency of classical definition but this definition is not suitable from mathematical view point, as calculation of probability using this method needs actual computation and numerical data. This definition will produce result only when the principle of statistical regularity is satisfied. The assumption that the condition of the experiment is reproducible may not be true if conditions changes over time.

AXIOMETRIC DEFINITION OF PROBABILITY:

Let S be a sample space and **Ω** be the class of all events (σ field). Let P be a real valued function define on **Ω,** then P is called **a** probability function, and P(A) is called the probability of the event A if the following axioms hold:

1. (Non Negetivity): $P(A) \geq 0 \quad , \forall A \in \Omega$
2. (Certainity) : For the certain event S P(S) = 1
3. For any two disjoint events A and B, $P(A \cup B) = P(A) + P(B)$

3a. For any infinite sequence of mutually disjoints events A_1, A_2, ---------

$$P(A_1 \cup A_2 \cup A_3 \cup \text{--------}) = P(A_1) + P(A_2) + P(A_3) + \text{-----}$$

If P satisfies the above axioms, then sample space S is called a probability space.

RULE OF PROBABILITY:

1. Rule of Addition:

Theorem of total probability: If A_1, A_2, ----, A_m are m-mutually exclusive events then the probability of A_1 or A_2 or ------ or A_m is the sum of the probabilities of A_1, A_2, ----, A_m ie.

$P(A_1 \cup A_2 \cup \text{-------} \cup A_m) = \sum_{i=1}^{m} P(A_i)$

Proof: Let a finite sample space S of equally likely outcomes of a random experiment consists of n(S) sample points of which $n(A_1)$ are favorable for the event A_1 and $n(A_2)$ are favorable for the event A_2.

Now, $P(A_1 \cup A_2)$

$$= \frac{n(A_1 \cup A_2)}{n(S)}$$

$$= \frac{n(A_1) + n(A_2)}{n(S)}$$ (Since A_1 and A_2 are mutually exclusive)

$$= \frac{n(A_1)}{n(S)} + \frac{n(A_2)}{n(S)}$$

$$= P(A_1) + P(A_2)$$

Let this be true for r mutually exclusive events A_1, A_2, ------, A_n. Therefore,

$$P(A_1 \cup A_2 \cup .... \cup A_r) = \sum_{i=1}^{r} P(A_i).$$

Suppose, A_1, A_2 , ...,A_r, A_{r+1} are r+1 mutually exclusive events of S.

Now,

$$P(A_1 \cup A_2 ..... \cup A_{r+1})$$
$$= P\{(A_1 \cup A_2 \cup ... \cup A_r) \cup A_{r+1}\}$$
$$= P(A_1 \cup A_2 \cup ... \cup A_r) + P(A_{r+1})$$
$$= P(A_1) + P(A_2) + ... + P(A_r) + P(A_{r+1})$$

Therefore, by mathematical induction, for any integer m

Note 1: When A_1, A_2, ...,A_m are mutually exclusive and exhaustive events then

$$A_1 \cup A_2 \cup .... \cup A_m = S \Rightarrow P(A_1 \cup A_2 \cup .... \cup A_m) = P(S) \Rightarrow \sum_{i=1}^{m} P(A_i) = 1$$

('13)Note 2: Since A and $\overline{A}$ are mutually exclusive and exhaustive events,

P(A)+P($\overline{A}$) =1 $\Rightarrow P(\overline{A}) = 1 - P(A)$

Note 3: For any two events A and B, (A∩ $\boldsymbol{B}$) and ($A \cap \overline{B}$)are mutually exclusive and exhaustive events. Also A= (A∩ $\boldsymbol{B}$) $\cup (A \cap \overline{B})$, therefore

$$P(A) = P(A \cap B) + P(A \cap \overline{B}) \Rightarrow P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

Also,

$$A-(A\cap B)=A\cap\overline{B}\Rightarrow P(A-(A\cap B))=P(A\cap\overline{B})=P(A)-P(A\cap B)$$

Note 4: By De Morgan's Law

$A^c \cup B^c = (A\cap B)^c \Rightarrow P((A\cap B)^c) = 1 - P(A\cap B)$

Note 5: If $\emptyset$ ***be impossible event and A be any event then A = A*** $\cup\ \emptyset$

$\Rightarrow P(A) = P(A) + P(\emptyset)$

$\Rightarrow P(\emptyset) = 0$

('13)Generalised Theorem on Total Probability:

If A and B are any two events on a sample space S then ,

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

Proof: By definition of probability

$P(A\cup B)$

$$\frac{n(A\cup B)}{n(S)}$$

$$= = \frac{n(A)+n(B)-n(A\cap B)}{n(S)}$$

$$= P(A)+P(B)-P(A\cap B)$$

Theorem ('06,'08): If A_1, A_2, A_3 are any three events then show that

$$P(A_1\cup A_2\cup A_3)=P(A_1)+P(A_2)+P(A_3)-P(A_1\cap A_2)-P(A_2\cap A_3)-P(A_3\cap A_1)+P(A_1\cap A_2\cap A_3)$$

Proof: We have, $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

Now, $P(A_1\cup A_2\cup A_3)$

$$= P(A_1\cup(A_2\cup A_3))$$
$$= P(A_1)+P(A_2\cup A_3)-P(A_1\cap(A_2\cup A_3))$$
$$= P(A_1)+P(A_2)+P(A_3)-P(A_2\cap A_3)-P((A_1\cap A_2)\cup(A_1\cap A_3))$$
$$= P(A_1)+P(A_2)+P(A_3)-P(A_2\cap A_3)-P(A_1\cap A_2)-P(A_1\cap A_3)+P(A_1\cap A_2\cap A_3)$$
$$= P(A_1)+P(A_2)+P(A_3)-P(A_1\cap A_2)-P(A_2\cap A_3)-P(A_3\cap A_1)+P(A_1\cap A_2\cap A_3)$$

Compound Probability: The probability of occurrence of two or more events simultaneously is termed as compound probability. If A_1, A_2, ..., A_n are events in a same sample space S, then their compound probability is denoted by $P(A_1\cap A_2\cap \ldots \cap A_n)$ or $P(A_1A_2 \ldots A_n)$. It is also known as joint probability.

Conditional Probability: If A and B are two events in the samplr space S of a random experiment such that P(A) $>$ 0. Now the probability of occurence of event B , subject to the condition that event A has already occured, is called the conditional probability of B , given A. It is denoted by P(B/A).

Theorem on compound Probability: The probability of simultaneous occurence of two events of a sample space S is equal to the product of the probability of one of the events by the conditional probability of the other; given that the first one has already occured , ie. if A and B are two events in the sample space S of a random experiment then,

P(AB) = P(A)P(B/A) = P(B)P(A/B)

Proof: Let S be a finite sample space of equally likely outcomes of random experiment and A and B are any two events in S.

If A has already occured then the number of sample points favorable for the event B = n(A∩B).

Therefore, P(B/A)

$$= \frac{n(A \cap B)}{n(A)}$$

$$= \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(A)}{n(S)}}$$

$$= \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A \cap B) = P(A)P(B/A) ----(1)$$

Similarly, if B is already occured then

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A \cap B) = P(B)P(A/B) ----(2)$$

From (1) and (2), P(AB)

= P(A)P(B/A) if P(A)$>$0

= P(B)P(A/B) if P(B) $>$0

Note: For three events A, B, C

P(ABC) = P((A∩B)∩C) = P(A∩B) P(C/ A∩B) = P(A)P(B/A)P(C/ A∩B)

Theorems Based on Axiomatic Approach:

Theorem 1: The impossible event $\emptyset$ has probability zero ie. $P(\emptyset) = 0$.

Proof: Let A be any event in the sample space S of a random experiment and $\emptyset$ be the impossible event in S. Now, we have

$A = A \cup \emptyset \Rightarrow P(A) = P(A \cup \emptyset) \Rightarrow P(A) = P(A) + P(\emptyset)$ [since A and $\emptyset$ are mutually exclusive]

$\Rightarrow P(\emptyset) = 0$.

('13)Theorem 2: For any event A, $P(A^c) = 1 - P(A)$.

Proof: Let A be any event in the sample space S. For the certain event S we know from axiom of probability that $P(S) = 1$. Now, we have

$A \cup A^c = S \Rightarrow P(A \cup A^c) = P(S) \Rightarrow P(A) + P(A^c) = 1$ [As A and A^c are mutually disjoint]

$\Rightarrow P(A^c) = 1 - P(A)$.

('09)Theorem 3 : If A_1 and A_2 are events in a probability space S such that $A_1 \subseteq A_2$ then $P(A_1) \le P(A_2)$ and hence $P(A_2 - A_1) = P(A_2) - P(A_1)$.

Proof: We have,

$$P(A_2) = A_1 \cup (A_2 - A_1)$$

$$\Rightarrow P((A_2) = P(A_1) + P(A_2 - A_1) \qquad \text{[Since } A_2 \text{ and } A_2 - A_1 \text{ are disjoint events]}$$

$$\Rightarrow P(A_2 - A_1) = P(A_2) - P(A_1)$$

Again, by axiom of probability we know that

$$P(A_2 - A_1) \ge 0 \Rightarrow P(A_2) - P(A_1) \ge 0 \Rightarrow P(A_2) \ge P(A_1).$$

Theorem 4: For any event A , $0 \le \boldsymbol{P(A)} \le \mathbf{1}$

Proof: Let A be any event in the probability space S. It is clear that $P(A) \ge \mathbf{0}$. Thus to prove the theorem we need to prove that $\boldsymbol{P(A)} \le \mathbf{1}$.

We have

$$S = A \cup (\boldsymbol{S} - \boldsymbol{A})$$

$$\Rightarrow P(S) = P(A \cup (S - A))$$

$$\Rightarrow 1 = P(A) + P(S - A)$$

$$\Rightarrow 1 - P(A) = P(S - A) \qquad \text{[Since } P(S)=1 \text{ and A and S-A are mutually disjoint events]}$$

$$\Rightarrow 1 - P(A) \ge 0$$

$$\Rightarrow P(A) \le 1$$

Thus we have $0 \le \boldsymbol{P(A)} \le \mathbf{1}$.

Theorem 5: If A and B are any two events then $\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$.

Proof: Let A and B be any two events in the probability space S. Now, we have

$$\boldsymbol{A} \cup \boldsymbol{B} = \boldsymbol{A} \cup [\boldsymbol{B} - \boldsymbol{A} \cap \boldsymbol{B}]$$

$\Rightarrow P(A \cup B) = P(A) + P(B - A \cap B)$ [Since A and B- (A$\cap$B) are disjoint events]

= P(A) + P(B) - P($A \cap B$) [Since P(B- (A$\cap$B)) = P(B) - P(A$\cap$B)]

Theorem 6 : For any two events A and B, P(A - B) = P(A) - P(A$\cap$B).

Proof: We have, A - B = A - (A$\cap$B)

$\Rightarrow P(A-B) = P(A-(A\cap B)) = P(A) - P(A\cap B)$ {AS, P(A - B) = P(A) -P(B)}

Conditional Probability:

Suppose E is an event in a sample space S with P(E) $>$ 0. Now the probability that an event A occurs once E has already occurred is called conditional probability of A given E. It is denoted by P(A/E).

Theorem: If A and B are two events in a equiprobable space S then $P(A/E) = \dfrac{P(A \cap E)}{P(E)}$.

Proof: Suppose s is an equiprobable space and n(E) and n(A) be the numbers of elements in the event E and A respectively.

$$\therefore P(E) = \frac{n(E)}{n(S)}, P(A \cap E) = \frac{n(A \cap E)}{n(S)}$$

Now,

P(A/E) = The relative probability of A w. r. t. the reduced space E.

$$= \frac{n(A \cap E)}{n(E)}$$

$$= \frac{n(A \cap E)/n(S)}{n(E)/n(S)}$$

$$= \frac{P(A \cap E)}{P(E)}$$

Independent Event: Two events A and B in a probability space S are said to be independent if the occurrence of one of them does not influence the occurrence of the other. More specifically, A and B are independent if P(A/B) = P(A) or P(B/A) = P(B). Therefore,

$$P(A \cap B) = P(A)P(B)$$

Note: Three events A, B and C are independent if the following two conditions are hold:

(1) $P(A \cap B) = P(A)P(B)$, $P(B \cap C) = P(B)P(C)$, $P(A \cap C) = P(A)P(C)$

(2) $P(A \cap B \cap C) = P(A)P(B)P(C)$.

Theorem of Total Probability: Let E be an event in a sample space S and let A_1, A_2, ---, A_n be mutually disjoint events whose union is S (ie. exhaustive), then

$$P(E) = \sum_{i=1}^{n} P(A_i)P(E/A_i)$$

Proof: Here, we have

$$S = \bigcup_{i=1}^{n} A_i, and, A_i \cap A_j = \phi for i \neq j, and, i, j \in \{1,2,.....,n\}$$

Now for any event E of S

$$E = E \cap S$$

$$= E \cap (\bigcup_{i=1}^{n} A_i)$$

$$= \bigcup_{i=1}^{n} (E \cap A_i)$$

$$\therefore P(E) = P(\bigcup_{i=1}^{n} (E \cap A_i)..........(1)$$

We claim that for $i \neq j$, $E \cap A_i$ and $E \cap A_j$ are disjoint, for if $(A \cap E_i) \cap (A \cap E_j) \neq \phi$ then there exist

$$x \in (A \cap E_i) \cap (A \cap E_j)$$

$$\Rightarrow x \in A, x \in E_i, and, x \in E_j$$

$$\Rightarrow x \in E_i \cap E_j$$

Which is a contradiction as E_i and E_j are disjoint.

$$\therefore P(\bigcup_{i=1}^{n} (E \cap A_i) = \sum_{i=1}^{n} P(E \cap A_i) = \sum_{i=1}^{n} P(A_i)P(E/A_i),$$ [by multiplication theorem.]

Thus from (1), we have $P(E)=\sum_{i=1}^{n}P(A_i)P(E/A_i)$

Baye's Theorem:

Let E be any event in a sample space S and A_1 , A_2 , ..., A_n be disjoint events whose union is S, then for any k = 1,2,....,n ,

$$P(A_k/E)=\frac{P(A_k)P(E/A_k)}{\sum_{i=1}^{n}P(A_i)P(E/A_i)}$$

Proof: By the theorem of total probability ,we have

$$P(A_k/E)=\frac{P(A_k\cap E)}{P(E)}$$

$$=\frac{P(A_k)P(E/A_k)}{P(E)}$$

$$=\frac{P(A_k)P(E/A_k)}{\sum_{i=1}^{n}P(A_i)P(E/A_i)}$$

('09) Ex. 1: A card is drawn at random from an ordinary deck of plying cards. Find the probability that (a) an ace, (b) a jack of hearth, (c) a three ob clubs or a six of diamonds, (d) a heart, (e) any suit except hearts, (f) a ten or a spade, (g) neither a four nor a club.

('11) Ex.2: A fair die is tossed twice. Find the probability of getting 4 , 5, or 6 on the first toss and 1, 2, 3, or 4 on the 2nd toss.

('07) Ex. 3: Find the probability of not getting a 7 or 11 total on either of two tosses of a pair of fair dice.

('11) Ex. 4: Two cards are drawn from a well-shuffled ordinary deck of 52 cards. Find the probability that they are both aces if the first card is (a) replaced, (b) not replaced.

('09) Ex. 5: Three balls are drawn at random from a box containing 6 red balls, 4 white balls, 5 blue balls. Find the probability that they are drawn in the order red, white , and blue if each ball is (a) replaced (b) not replaced.

('06) Ex. 6: For each n+1 events $A_0, A_1, A_2, \ldots, A_n$ such that

$P(A_0A_1 \ldots A_n) > 0$. Prove that

$$P(A_0A_1 \ldots A_n) = P(A_0)\, P(A_1/A_0)\, P(A_2/A_0A_1) \ldots\ldots\ldots P(A_n/A_0A_1\ldots\ldots A_{n-1}).$$

Ex. 7: There are three bags first containing 1 white, 2 red, 3 green balls second contains 2 white, 3 red , 1 green ball and the third contains 3 white , 1 red, 2 green balls. Two balls are . drawn from a beg chosen at random , there are found to be one white and one red. Show that the probability that the ball so drawn come from the second beg is 6/11.

('10) Ex 8: For the two mutually exclusive events A and B , P(B) = 2 P(A) and $A \cup B = S$, find P(A).

('07) Ex. 9: Three machines M_1, M_2, M_3 produce identical items. Of their respective output 5%, 4% and 3% of items are faulty. On a certain day M_1 has produced 30% and M_3 the remainder. An item selected at random is found to be faulty. Applying Baye's theorem show that the chance that it was produced by the machine with the largest output is 0.355.

('08) Ex. 10: If an event A must result in one of mutually exclusive events A_1, A_2, A_3 then prove that

$$P(A) = P(A_1)\, P(A/A_1) + P(A_2)\, P(A/A_2) + P(A_3)\, P(A/A_3).$$

Ex. 11: India plays 2, 3, and 5 matches against Pakistan, Sri Lanka and West Indies respectively. Probability that India wins a match against Pakistan, Sri Lanka and West Indies are respectively 0.6, 0.5, and 0.4. If India wins a match, show that the probability that it was against Pakistan is 12/47.

('08) Ex. 12: A bag contains n balls, one of which is white. The probability that A and B soeak truth is p_1 and p_2 respectively. One ball is drawn from the bag and A and B both assert that it is white. Show that the probability that the drawn ball is actually white is

$$\frac{(n-1)p_1p_2}{(n-1)p_1p_2 + (1-p_1)(1-p_2)}$$

Ex. 13: In a certain college there were three candidates for the position of principal, Dr. Chaliha, Dr. Das and Dr. Hazarika whose chances of getting the appointment are in the propoortion 4:2:3 respectively. The probability that Dr. Chaliha if selected would introduce post graduate classes in the college is 0.3 and that of Dr. Das and Dr. Hazarika are 0.5 and 0.8 respectively.

(i) Find the probability of introducing post graduate classes in the college.

(ii) Find the probability that Dr. Chaliha will be selected as principal given that P.G. classes were introduce in the college.

('10) Ex. 14: Prove the following laws assuming that in each case the conditional probability are defined:

(i) P(B) = 1 then P(A/B)= P(A)

(ii) If A and B are disjoint and P(B) > 0, then P(A/B) = 0.

('11) Ex. 15: Given P(A) = 0.30 , P(B) = 0.78 , P(A∩B) = 0.16. Find $P(A\cap\overline{B}), P(A/\overline{B}), P(\overline{B}/A)$.

('06) Ex. 16: A bag contains 40 tickets numbered 1,2,3, ..., 40 of which 4 are drawn at random and arranged in ascending order ($t_1 < t_2 < t_3 < t_4$). Show that the probability of t_3 being 25 is 414/9139.

('13) Ex17: A bag contains 6 white and 9 black balls. four balls are drawn at a time. Find the probability for the first draw to give 4 white and the second draw to give 4 black balls if the balls are not replaced before the 2nd draw. 5

('13) Ex 18: If A ∩ B = ϕ, then show that P(A) $\leq \boldsymbol{P}(\overline{\boldsymbol{B}})$. 3

('13) Ex. 19: The contents of urns I, II and III are as follows:

1 white, 2 black and 3 red balls ; 2 white, 1 black and 1 red balls ; 4 white, 5 black and 3 red balls. One urn is chosen at random and two balls are drawn. They happen to be white and red. What is the probability that they come from urn I , II or III. 7